5.6 Tensors

The last formal mathematical tool that we need to introduce is the concept of a tensor. A tensor is a generalization of the idea of a four-vector, and as such a tensor represents geometrical object existing in spacetime, but one that is even more difficult to visualize than a four-vector.

One viewpoint with regard to tensors is that they can be considered as being 'operators' that act upon four-vectors to produce real numbers, and that is the way that the concept will be introduced here. The connection between tensors defined in this manner, and concepts already introduced will emerge later, as will the physical applications of the idea.

Thus, we begin with a definition.

> *A tensor* $\mathsf{T}(\vec{a}, \vec{b}, \vec{c}, \ldots)$ *is a linear function of the four-vectors* $\vec{a}, \vec{b}, \vec{c}, \ldots$ *that maps these four-vectors into the real numbers.*

Different rules for how the real number is calculated from the vector arguments then gives rise to different tensors. The manner of definition, namely that no mention is made of any reference frame, means that a tensor is a quantity that is independent of the choice of reference frame.

The following properties and definitions are important:

Rank The *rank* of a tensor is the number of vector arguments. Thus a tensor of rank 1 will be the function of one vector only, i.e. $\mathsf{p}(\vec{a})$ will define a tensor of rank one, $\mathsf{g}(\vec{a}, \vec{b})$, a tensor of rank 2 and so on.

Linearity That the function $\mathsf{T}(\vec{a}, \vec{b}, \vec{c}, \ldots)$ is linear means that for any numbers u and v

$$\mathsf{T}(u\vec{a} + v\vec{b}, \vec{c}, \ldots) = u\mathsf{T}(\vec{a}, \vec{c}, \ldots) + v\mathsf{T}(\vec{b}, \vec{c}, \ldots) \tag{5.83}$$

with same being true for all the arguments, i.e.

$$\mathsf{T}(\vec{a}, u\vec{b} + v\vec{c}, \ldots) = u\mathsf{T}(\vec{a}, \vec{b}, \ldots) + v\mathsf{T}(\vec{a}, \vec{c}, \ldots). \tag{5.84}$$

Tensor Components The components of a tensor are the values of the tensor obtained when evaluated for the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, ... equal to the basis vectors. Thus, we have

$$T_{\mu\nu\alpha\ldots} = \mathsf{T}(\vec{e}_\mu, \vec{e}_\nu, \vec{e}_\alpha, \ldots). \tag{5.85}$$

As a consequence of this and the linearity of T, we have

$$\mathsf{T}(\vec{a}, \vec{b}, \vec{c}, \ldots) = \mathsf{T}(\vec{e}_\mu, \vec{e}_\nu, \vec{e}_\alpha, \ldots)a^\mu b^\nu c^\alpha \ldots = T_{\mu\nu\alpha\ldots}a^\mu b^\nu c^\alpha \ldots. \tag{5.86}$$

Raising and Lowering Indices The process of raising and lowering indices can be carried through with the components of a tensor in the expected way. Thus, we can write

$$T^\mu{}_{\nu\alpha\ldots} = g^{\beta\mu} T_{\beta\nu\alpha\ldots} \tag{5.87}$$

or

$$T_\mu{}^\nu{}_\alpha \ldots = g^{\beta\nu} T_{\mu\beta\alpha\ldots}. \tag{5.88}$$

Corresponding to this we would have, for instance

$$\mathsf{T}(\vec{a}, \vec{b}, \vec{c}, \ldots) = T_{\mu\nu\alpha\ldots}a^\mu b^\nu c^\alpha \ldots = T_{\mu\nu\alpha\ldots}g^{\beta\mu} a_\beta b^\nu c^\alpha \ldots. \tag{5.89}$$

Where the implied sum over β means that we are applying the raising procedure to a_β. But, if we reqroup the terms, we have

$$\mathsf{T}(\vec{a}, \vec{b}, \vec{c}, \ldots) = g^{\beta\mu} T_{\mu\nu\alpha\ldots} a_\beta b^\nu c^\alpha \ldots \tag{5.90}$$

where we now see that the implied sum over μ means that we are raising an index in the components of the tensor, i.e.

$$\mathsf{T}(\vec{a}, \vec{b}, \vec{c}, \ldots) = T^\beta{}_{\nu\alpha\ldots} a_\beta b^\nu c^\alpha \ldots \tag{5.91}$$

This flexibility in moving indices up and down by the application of $g^{\mu\nu}$ or $g_{\mu\nu}$ means that we can express the components of any tensor in a number of ways that differ by the position of the indices. The different ways in which this is done is described by different terminology, i.e.

$T_{\mu\nu\alpha\ldots}$	covariant components of T
$T_\mu{}^\nu{}_{\alpha\ldots}$ or $T^{\mu\nu}{}_{\alpha\ldots}$ and other combinations of up and down indices	mixed components of T
$T^{\mu\nu\alpha\ldots}$	contravariant components of T.

Being able to raise and lower indices of a tensor raises the possibility of introducing a further mathematical manipulation of tensors. We will illustrate it in the case of a tensor of rank 2, $\mathsf{T}(\vec{a}, \vec{b})$ with covariant components $T_{\mu\nu}$ and mixed components $T_\mu{}^\beta$ where

$$T_\mu{}^\beta = g^{\beta\nu} T_{\mu\nu}. \tag{5.92}$$

If we set $\mu = \beta$ in the tensor component $T_\mu{}^\beta$, we obtain $T_\mu{}^\mu$ which, according to the summation convention implies a sum must be taken over the repeated index μ i.e.

$$T_\mu{}^\mu = T_0{}^0 + T_1{}^1 + T_2{}^2 + T_3{}^3. \tag{5.93}$$

This procedure is known as a *contraction* of the tensor. The effect of contraction of a tensor is to lower the rank of the tensor by 2, as seen here where the result is a number (a scalar), a tensor of rank 0.